Lecture 17

Fuzzy Measures - I

Belief Measure,
Plausibility Measure,
Basic Probability Assignment, and
Joint Basic Assignment
Types of Uncertainty

- Vagueness: associated with the difficulty of making sharp or precise distinctions in the world, i.e. fuzziness
  → Fuzzy sets

- Ambiguity: associated with one-to-many relations, i.e. difficult to make a choice between two or more alternatives, i.e. nonspecificity, conflict or dissonance, and confusion
  → Fuzzy measures
Ex) Criminal trial: The jury members are uncertain about the guilt or innocence of the defendant.

- Two crisp set:
  1) The set of people who are guilty of the crime
  2) The set of innocent people

- The concern:
  - Not with the degree to which the defendant is guilty, but with the degree to which the evidence proves his/her membership either in the crisp set of guilty people or in the crisp set of innocent people.
  - Our evidence is rarely, if ever, perfect, and some uncertainty usually prevails.
- Fuzzy measure:
  - To represent this type of uncertainty
  - Assign a value to each possible crisp set to which the element in question might belong, signifying the degree of evidence or belief that a particular element belongs in the set.
  - The degree of evidence, or certainty of the element’s membership in the set
**Fuzzy Measure vs. Fuzzy Set**

*Fuzzy Set:* \( x \in X \rightarrow \mu_A(x) \in [0,1] \)
where \( A \) is a fuzzy set, \( X \): universal set.

*Fuzzy Measure:* \( g(A_i) \in [0,1] \)
\( A_i \subseteq X \)
where \( A_i \) is a crisp set, \( X \): universal set.

Note that \( \mu_A : x \rightarrow [0,1] \)
\( g : P(X) \rightarrow [0,1] \)
where \( P(X) \) is a power set of \( X \).
**Fuzzy Measure**

*Def:* A FM \( g(.) \) is defined by a set function:

\[
g : P(X) \rightarrow [0,1]
\]

that satisfies the following axioms:

**g1:** Boundary condition

\[
g(\emptyset) = 0, \\
g(X) = 1 \rightarrow \text{The element in question belongs to the universal set } X.
\]

**g2:** Monotonicity

For every \( A, B \in P(X) \), if \( A \subseteq B \),
then \( g(A) \leq g(B) \)
g3: Continuity

For every sequence \((A_i \in P(X) | i \in N)\) of subset of \(X\), if either \(A_1 \subseteq A_2 \subseteq \cdots\) or \(A_1 \supseteq A_2 \supseteq \cdots\) (i.e. sequence is monotonic), then

\[
\lim_{i \to \infty} g(A_i) = g(\lim_{i \to \infty} A_i) = g(A),
\]

where \(A = \lim_{i \to \infty} A_i\), \(X\) : an infinite universal set.
A FM is often defined more generally as a function:
\[ g : \Gamma \rightarrow [0, 1], \]
where \( \Gamma \subset P(X) \) is a family of subsets of \( X \) and is a \emph{Borel field} or a \emph{\( \sigma \) field}.

**Def: Borel field \( \Gamma \)**

\( \Gamma \subset P(X) \) is a family of subset of \( X \):

1. \( \emptyset \in \Gamma \) and \( X \in \Gamma \)
2. If \( A \in \Gamma \), then \( \bar{A} \in \Gamma \).
3. \( \Gamma \) is closed under the operation of set union.

(i.e. \( A \in \Gamma \) and \( B \in \Gamma \), then \( A \cup B \in \Gamma \))
Remarks: \( g(A) \leq g(A \cup B) \), and \( g(B) \leq g(A \cup B) \)

(a) Since \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \) (by monotonic property \( g2 \)),

\[
\max[g(A), g(B)] \leq g(A \cup B)
\]

(b) Since \( A \cap B \subseteq A \) and also \( A \cap B \subseteq B \), \( g(A \cap B) \leq \min[g(A), g(B)] \) \( \text{ (g2) } \)

(c) The FM was introduced by Sugeno (’77) to exclude the “additivity” requirement of the standard measure \( \mu \), i.e.

if \( A \cap B = \phi \), then \( \mu(A \cup B) = \mu(A) + \mu(B) \).
Ex) Consider 3 poles/sticks of length 1, 2, and 3 (in or m).

Let “length” measure be $g(.)$. Then,

$$g(length = 3) \geq g(length = 1) + g(length = 2),$$

or

$$g(length = 3) < g(length = 1) + g(length = 2)$$

depending on how to measure because the intersection of $g(length=1)$ and $g(length=2)$ is $\phi$. 

Fuzzy Measure
Belief measure is a function such that $Bel: P(X) \rightarrow [0, 1]$ that satisfies axioms $g1$ to $g3$ and an additional $g4$ (subadditivity axiom):

\[
g4: Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_{i} Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) + \cdots + (-1)^{n-1} Bel(A_1 \cap A_2 \cap \cdots \cap A_n)
\]

for every $n \in N$ and for every collection of subsets of $X$. 
Belief Measure

Ex) \( n = 2, \)
\[
Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2)
\]

Let \( A_1 = A, \ A_2 = \overline{A} \)
\[
Bel(A \cup \overline{A}) = Bel(X) \geq Bel(A) + Bel(\overline{A}) - Bel(A \cap \overline{A})
\]

This leads to Lower Probability:
\[
Bel(A) + Bel(\overline{A}) \leq 1
\]
Plausibility Measure

Associated with belief measure is a plausibility measure defined as (Schafer, ’76)

\[ Pl(A) = 1 - Bel(\overline{A}) \quad \text{for all } A \in P(X). \]

Similarly,

\[ Bel(A) = 1 - Pl(\overline{A}). \]

- \( Bel(\cdot) \) and \( Pl(\cdot) \) are mutually dual.

**Def:** Plausibility measure is a function such that \( Pl: P(X) \rightarrow [0, 1] \) that satisfies axioms g1 to g3 and an additional g5:

\[ g5: \quad Pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_{i} Pl(A_i) - \sum_{i < j} Pl(A_i \cup A_j) + \cdots + (-1)^{n-1} Pl(A_1 \cup A_2 \cup \cdots \cup A_n) \]

for every \( n \in N \) and every collection of subsets of \( X \).
For $n = 2$, 

$$Pl(A_1 \cap A_2) \leq Pl(A_1) + Pl(A_2) - Pl(A_1 \cup A_2)$$

Let $A_1 = A, \ A_2 = \overline{A}$,

$$Pl(A \cap \overline{A}) \leq Pl(A) + Pl(\overline{A}) - Pl(A \cup \overline{A})$$

This lead to *Upper Probability*:

$$Pl(A) + Pl(\overline{A}) \geq 1$$
Basic Probability Assignment

Express both Belief $M.$ and its dual Plausibility $M.$ in terms of another set function $m(\cdot)$:

$$m : P(X) \rightarrow [0, 1]$$

such that

1. $m(\emptyset) = 0$
2. $\sum_{A \in P(X)} m(A) = 1$

Remarks:

(a) $m(A)$ is interpreted as the degree of evidence supporting the claim that a specific element $x \in X$ belongs to the set $A$ only and not to any special subset of $A$.

$m(\cdot)$ is called basic probability assignment (or basic assignment).
(b) It is not required that \( m(X) = 1 \) : g1
(c) It is not required that
\[
m(A) \leq m(B) \quad \text{when } A \subseteq B
\]
: g2
(d) No relationship between \( m(A) \) and \( m(\bar{A}) \)
\[
\Rightarrow m(A) + m(\bar{A}) \nleq 1
\]
\[
\leq 1
\]
(e) \( m(\cdot) \) is not a fuzzy measure.
(f) \( Bel(\cdot) \) and \( Pl(\cdot) \) can be determined from \( m(\cdot) \) uniquely:
\[
Bel(A) = \sum_{B \subseteq A} m(B)
\]
\[
Pl(A) = \sum_{B \cap A \neq \phi} m(B) \quad \text{for all } A \in P(X)
\]
Distinguish among $m(A)$, $Bel(A)$, and $Pl(A)$.

- $m(A)$: the *degree of evidence* or belief that the element $x$ belongs to the set $A$ alone.

- $Bel(A)$: the total evidence or belief that $x$ belongs to the set $A$ as well as to the various subsets of $A$.

- $Pl(A)$: not only the total evidence that $x$ belongs to the set $A$ and its subsets of set $A$ but also the additional evidence or belief associated with sets that overlap with set $A$. 
Ex) Let the universal set $X$ denote the set of all possible diseases $P$: pneumonia, $B$: bronchitis, $E$: emphysema

<table>
<thead>
<tr>
<th></th>
<th>$m$</th>
<th>$Bel$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
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<td>0.05</td>
</tr>
<tr>
<td>$B$</td>
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<td>0</td>
</tr>
<tr>
<td>$E$</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>$P \cup B$</td>
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<td>0.2</td>
</tr>
<tr>
<td>$P \cup E$</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>$B \cup E$</td>
<td>0.05</td>
<td>0.1</td>
</tr>
<tr>
<td>$P \cup B \cup E$</td>
<td>0.6</td>
<td>1</td>
</tr>
</tbody>
</table>

$Bel(A) = \sum_{B \subseteq A} m(B)$

where $B$: all the possible subset of $A$

$\rightarrow Bel(X) = 1$: boundary condition $(\because X = P \cup B \cup E)$
Bel(P) = m(P) = 0.05

\[ Bel(P \cup B) = m(P \cup B) + m(P) + m(B) = 0.15 + 0.05 + 0 = 0.2 \]

\[ Pl(P \cup B) = m(P \cup B) + m(P) + m(B) + m(P \cup E) + m(B \cup E) + m(P \cup B \cup E) \]
\[ = 0.15 + 0.05 + 0 + 0.1 + 0.05 + 0.6 = 0.95 \]
Given $Bel(\cdot)$, find $m(\cdot)$:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} Bel(B) \quad \forall A \in P(X)$$

where $|A-B|$ is the size of $(A-B)$, i.e. the cardinality of crisp set $(A-B)$

Ex) $m(P) = (-1)^{|\phi|} Bel(P)$

$$= (-1)^0 Bel(P) = 0.05 \quad (\because |P - P| = |\phi| = 0)$$

$$m(P \cup B) = (-1)^{|\phi|} Bel(P \cup B) + (-1)^{|P|} Bel(B)$$

$$+ (-1)^{|B|} Bel(P)$$

$$= 0.2 - 0 - 0.05 = 0.15$$
Basic Probability Assignment

Def: Every \( A \in P(X) \) for which \( m(A) > 0 \) is called a focal element (cf. “support” in fuzzy set)

- Focal elements are subset of \( X \) on which the available evidence focuses.
- When \( X \) is finite, \( m \) can be characterized by a list of focal elements \( A \) and its corresponding values of \( m(A) \).
- Form a body of evidence as an ordered pair \((F, m)\), where \( F \) denotes a set of focal elements and \( m \) its associated values of basic assignment.
- Total ignorance is expressed in terms of \( m(\cdot) \) by
  \[
  m(X) = 1, \quad m(\emptyset) = 0, \quad \text{and} \quad m(A) = 0 \quad \text{for all} \ A \neq X
  \]
Def: A basic assignment $m$ is said to be “simple support function” focused at $A$ if

1. $m(A) = s$
2. $m(X) = 1 - s$
3. $m(B) = 0$ for all other sets, where $B \in P(X)$. 

Basic Probability Assignment

<table>
<thead>
<tr>
<th></th>
<th>$m(A)$</th>
<th>$Bel(A)$</th>
<th>$Pl(A)$</th>
<th>$Bel(\overline{A})$</th>
<th>$Pl(\overline{A})$</th>
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</thead>
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<td>0.05</td>
<td>0.9</td>
<td>0.1</td>
<td>0.95</td>
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<tr>
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</tr>
<tr>
<td>$E$</td>
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<td>0.05</td>
<td>0.8</td>
<td>0.2</td>
<td>0.95</td>
</tr>
<tr>
<td>PUB</td>
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<td>0.95</td>
<td>0.05</td>
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<tr>
<td>PUE</td>
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<td>0.2</td>
<td>1</td>
<td>0</td>
<td>0.8</td>
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<tr>
<td>BUE</td>
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<td>0.1</td>
<td>0.95</td>
<td>0.05</td>
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</tr>
<tr>
<td>PUBUE</td>
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<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$$\sum m = 1 \quad Bel(X) = 1 \quad (\because X = P \cup B \cup E)$$

Note: $Bel(A) + Bel(\overline{A}) \leq 1$

$Bel(A) = \sum_{B \subseteq A} m(B), \quad Pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$

$Bel(P \cup E) = m(P \cup E) + m(P) + m(E) = 0.1 + 0.05 + 0.05 = 0.2$

$Pl(P \cup E) = m(P \cup E) + m(P) + m(E) + m(P \cap B) + m(B \cup E) + m(P \cap B \cup E)$

$$= 0.1 + 0.05 + 0.05 + 0.15 + 0.05 + 0.6 = 1$$

$Pl(A) = 1 - Bel(\overline{A}) \quad \Rightarrow \quad Bel(\overline{A}) = 1 - Pl(A)$

$Bel(A) = 1 - Pl(\overline{A}) \quad \Rightarrow \quad Pl(\overline{A}) = 1 - Bel(A)$
Given two basic assignments $m_1$ and $m_2$ on $P(X)$ which are evidences from two independent sources. Find joint basic assignment $m_{1,2}(A)$:

$$m_{1,2}(A) = \frac{\sum_{B \cap C = A} m_1(B) \cdot m_2(C)}{1 - K}$$

All the possible subsets

Normalization factor

for $A \neq \phi$,

where

$$K = \sum_{B \cap C = \phi} m_1(B) \cdot m_2(C)$$

and $m_{1,2}(\phi) = 0$.

→ Dempster’s rule of combination

(or Dempster-Shafer Theory, Shafer ’76, Pearl ’88)
### Joint Basic Assignment

<table>
<thead>
<tr>
<th>Ex) Focal elements</th>
<th>Expert1</th>
<th>Expert2</th>
<th>Combined evidence</th>
</tr>
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<tr>
<td></td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$m_{1,2}$</td>
</tr>
<tr>
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<td>0.3</td>
<td></td>
</tr>
<tr>
<td>$B$</td>
<td>0.3</td>
<td>0.4</td>
<td>?</td>
</tr>
<tr>
<td>$A \cup B$</td>
<td>0.6</td>
<td>0.3</td>
<td></td>
</tr>
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</table>

#### i) Calculate $1-K$

$$1 - \sum_{B \cap C = \emptyset} m_1(B)m_2(C) = 1 - m_1(A)m_2(B) - m_1(B)m_2(A) = 0.87$$

#### ii) $\sum_{B \cap C = A} m_1(B) \cdot m_2(C) / 1 - K$

$$m_{1,2}(A) = \frac{1}{0.87} \left[ m_1(A)m_2(A) + m_1(A)m_2(A \cup B) + m_1(A \cup B)m_2(A) \right] = 0.276$$

$$m_{1,2}(B) = \frac{1}{0.87} \left[ m_1(B)m_2(B) + m_1(B)m_2(B \cup C) + m_1(A \cup B)m_2(B) \right] = 0.517$$

$$m_{1,2}(A \cup B) = \frac{1}{0.87} \left[ m_1(A \cup B)m_2(A \cup B) \right] = 0.207$$
Ex) Assume someone discovers an old painting by Raphael.

Question 1) Painting done by Raphael ($R$)?
2) Painting done by his disciples ($D$)?
3) Painting is a counterfeit ($C$)?

$$X = R \cup D \cup C$$

<table>
<thead>
<tr>
<th>$A$</th>
<th>$m_1$</th>
<th>Bel$_1$</th>
<th>$m_2$</th>
<th>Bel$_2$</th>
<th>$m_{1,2}$</th>
<th>Bel$_{1,2}$</th>
<th>Pl$_{1,2}$</th>
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<td>0.05</td>
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<td>0.15</td>
<td>0.21</td>
<td>0.21</td>
<td>0.84</td>
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<tr>
<td>$D$</td>
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<td>0</td>
<td>0</td>
<td>0.01</td>
<td>0.01</td>
<td>0.5</td>
</tr>
<tr>
<td>$C$</td>
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<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
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<td>0.09</td>
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<td>0.2</td>
<td>0.4</td>
<td>0.2</td>
<td>0.5</td>
<td>0.99</td>
</tr>
<tr>
<td>$D \cup C$</td>
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<td>0.1</td>
<td>0.05</td>
<td>0.1</td>
<td>0.06</td>
<td>0.16</td>
<td>0.79</td>
</tr>
<tr>
<td>$R \cup D \cup C$</td>
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<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0.31</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$$\sum m_{1,2} = 1$$
Joint Basic Assignment

All the possible subsets with intersection $= \phi$:

$$1 - K = 1 - [m_1(R)m_2(D) + m_1(R)m_2(C) + m_1(R)m_2(D \cup C) + m_1(D)m_2(R) + m_1(D)m_2(C) + m_1(D)m_2(R \cup C) + m_1(C)m_2(R) + m_1(C)m_2(D) + m_1(C)m_2(R \cup D) + m_1(R \cup D)m_2(C) + m_1(R \cup C)m_2(D) + m_1(D \cup C)m_2(R)] = 0.97$$

$$m_{1,2}(R) = [m_1(R)m_2(R) + m_1(R)m_2(R \cup D) + m_1(R)m_2(R \cup C) + m_1(R)m_2(R \cup D \cup C) + m_1(R \cup D)m_2(R) + m_1(R \cup D)m_2(R \cup C) + m_1(R \cup C)m_2(R) + m_1(R \cup C)m_2(R \cup D) + m_1(R \cup D \cup C)m_2(R)] / 0.97 = 0.21$$

$$m_{1,2}(R \cup D \cup C) = m_1(R \cup D \cup C)m_2(R \cup D \cup C) / 0.97 = 0.31$$