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Direct Model Reference Adaptive Pole Placement Control with Exponential Weighting Properties

Jong-Hwan Kim, Yeon-Chan Hong, and Keh-Kun Choi

Abstract—**A parameterization for a linear system is presented to design a direct model reference adaptive pole placement controller. This parameterized model is one of the structured nonminimal models. The exponentially weighted least-squares algorithm is employed to estimate the control parameters. The direct adaptive controller has the exponential weighting properties by the proposed method of selecting the characteristic polynomials of the sensitivity function filters in connection with the reference model.**

I. Introduction

The parameterization is the presentation of the system model through the control parameters so that these parameters can be directly estimated. The problem of parameterization for direct adaptive control has been studied in the literature [1]-[5]. Recently, Heymann [5] generalized the parametrization for direct adaptive control with the structured nonminimal models, and presented persistency of excitation conditions. The parameterized model of this note which is derived from the configuration of Fig. 1 is one of the above structured nonminimal models, and is used to design a direct model reference adaptive pole placement controller.

This note focuses on the problem of exponential data weighting in direct adaptive control. It is the case where the most recent data are assumed to be more informative than past data and hence old data are exponentially discarded. The key idea is philosophically the same as that of [6]. To estimate the control parameters, an exponentially weighted least-squares algorithm is employed. By this algorithm, the past measurement vectors can be weighted exponentially, however, the past measurement data in those vectors may not be weighted exponentially for some zeros of the characteristic polynomials of the sensitivity function filters [3]. In order to maintain the exponential weighting properties of the algorithm for all the past measurement data, a scheme of selecting the stable characteristic polynomials of the sensitivity function filters in connection with the reference model is proposed. Since all the past measurement data can be weighted exponentially, this proposed scheme guarantees fast convergence of all estimates.

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J.-H. Kim is with the Department of Electrical Engineering, Korea Advanced Institute of Science and Technology, Taejon-shi, 305-701, Republic of Korea and a Research member of ERC-ACI by KOSEF.

Y.-C. Hong is with the Department of Electronics Engineering, Soonchunhyang University, Chungnam 337-880, Republic of Korea.

K.-K. Choi is with the Department of Electronics Engineering, Seoul National University, Seoul 151, Republic of Korea.

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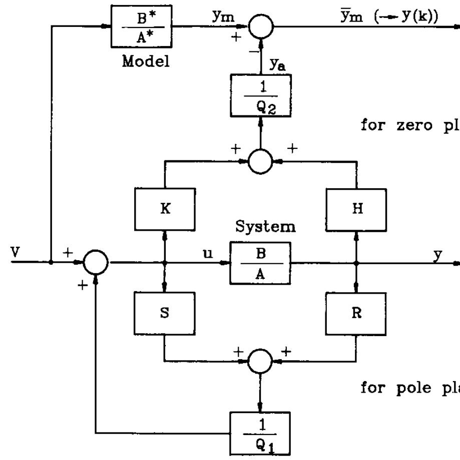

Fig. 1. Control structure.

II. Direct Model Reference Adaptive Pole Placement

The system to be controlled is assumed to be represented by the following discrete-time model:

$$A(q^{-1})z(k) = u(k)$$
$$y(k) = B(q^{-1})z(k) \tag{1}$$

where $u(k)$ and $y(k)$ are the measurable system input and output, respectively, $z(k)$ is the internal state variable, and $A(q^{-1})$ and $B(q^{-1})$ are polynomials in the unit delay operator q^{-1}, of the form

$$A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_rq^{-r}$$
$$B(q^{-1}) = b_1q^{-1} + \cdots + b_rq^{-r}.$$

We assume that the order r is known and the system model (1) is minimal, that is, the polynomials $A(q^{-1})$ and $B(q^{-1})$ are coprime. Consider the fixed control strategy

$$Q_1(q^{-1})u(k) = R(q^{-1})y(k) + S(q^{-1})u(k) + Q_1(q^{-1})v(k) \tag{2}$$

where $v(k)$ is an external reference signal, $R(q^{-1})$ and $S(q^{-1})$ are controller polynomials, and $Q_1(q^{-1})$ is an arbitrary stable polynomial, of the form

$$R(q^{-1}) = r_1q^{-1} + \cdots + r_rq^{-r}$$
$$S(q^{-1}) = s_1q^{-1} + \cdots + s_rq^{-r}$$
$$Q_1(q^{-1}) = q_{10} + q_{11}q^{-1} + \cdots + q_{1r}q^{-r}, \qquad q_{10} = 1.$$

Applying this control law to the system, the pole placement equation can be derived as follows:

$$A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = Q_1(q^{-1})[A(q^{-1}) - A^*(q^{-1})] \tag{3}$$

where $A^*(q^{-1})$ is a predetermined monic stable polynomial of the form

$$A^*(q^{-1}) = a_0^* + a_1^*q^{-1} + \cdots + a_r^*q^{-r}, \qquad a_0^* = 1.$$

Thus the closed-loop system becomes

$$y(k) = \frac{B(q^{-1})}{A^*(q^{-1})}v(k). \tag{4}$$

The actual system is controlled as in Fig. 1. The lower part is used for the pole placement. The upper part is used for the zero placement to represent the closed-loop system in terms of the reference model and the auxiliary signal.

From the configuration of Fig. 1, the augmented model output $\bar{y}_m(k)$ is defined as follows:

$$\bar{y}_m(k) = y_m(k) - y_a(k) \tag{5}$$

where $y_m(k)$ is the reference model output, and $y_a(k)$ is the auxiliary signal. Then $\bar{y}_m(k)$ can be obtained as

$$\bar{y}_m(k) = \frac{Q_2 B^* - AK - BH}{A^*(q^{-1})Q_2(q^{-1})} v(k). \tag{6}$$

From (6), the zero placement equation is defined as follows:

$$A(q^{-1})K(q^{-1}) + B(q^{-1})H(q^{-1}) = Q_2(q^{-1})[B^*(q^{-1}) - B(q^{-1})] \tag{7}$$

where $H(q^{-1})$ and $K(q^{-1})$ are auxiliary polynomials, and $Q_2(q^{-1})$ and $B^*(q^{-1})$ are stable polynomials, of the form

$$H(q^{-1}) = h_1 q^{-1} + \cdots + h_r q^{-r}$$
$$K(q^{-1}) = k_1 q^{-1} + \cdots + k_r q^{-r}$$
$$B^*(q^{-1}) = b_1^* q^{-1} + \cdots + b_r^* q^{-r}$$
$$Q_2(q^{-1}) = q_{20} + q_{21} q^{-1} + \cdots + q_{2r} q^{-r}, \qquad q_{20} = 1.$$

When (7) holds, the augmented model output $\bar{y}_m(k)$ becomes the closed-loop system output $y(k)$.

$$\bar{y}_m(k) = \frac{B(q^{-1})}{A^*(q^{-1})} v(k) = y(k). \tag{8}$$

From pole and zero placement equations (3), (7), and (1), we get

$$F(q^{-1}) y(k) = G(q^{-1}) u(k) \tag{9}$$

where

$$F(q^{-1}) = Q_1(q^{-1})Q_2(q^{-1})A^*(q^{-1}) + Q_2(q^{-1})B^*(q^{-1})R(q^{-1}) + Q_1(q^{-1})A^*(q^{-1})H(q^{-1})$$
$$G(q^{-1}) = Q_1(q^{-1})Q_2(q^{-1})B^*(q^{-1}) - Q_2(q^{-1})B^*(q^{-1})S(q^{-1}) - Q_1(q^{-1})A^*(q^{-1})K(q^{-1}).$$

The above system model is a specially structured nonminimal model [5]. In this case, $F(q^{-1}) = L(q^{-1})A(q^{-1})$ and $G(q^{-1}) = L(q^{-1})B(q^{-1})$ are derived for the following common factor $L(q^{-1})$

$$L(q^{-1}) = Q_1(q^{-1})Q_2(q^{-1}) + Q_1(q^{-1})H(q^{-1}) - Q_2(q^{-1})S(q^{-1}) - S(q^{-1})H(q^{-1}) + R(q^{-1})K(q^{-1}).$$

The following design identity may be satisfied by the coprimeness assumption for $A(q^{-1})$ and $B(q^{-1})$ [5].

$$[Q_1(q^{-1})Q_2(q^{-1})A^*(q^{-1}) + Q_2(q^{-1})B^*(q^{-1})R(q^{-1}) + Q_1(q^{-1})A^*(q^{-1})H(q^{-1})]B(q^{-1})$$
$$= [Q_1(q^{-1})Q_2(q^{-1})B^*(q^{-1}) - Q_2(q^{-1})B^*(q^{-1})S(q^{-1}) - Q_1(q^{-1})A^*(q^{-1})K(q^{-1})]A(q^{-1}). \tag{10}$$

Therefore, (9) with $F(q^{-1})$ and $G(q^{-1})$ can be regarded as a parameterization of the system (1). The polynomials $Q_1(q^{-1})$ and $Q_2(q^{-1})$ are characteristic polynomials of the sensitivity function filters [3]. Equation (9) can be written more compactly as

$$y^*(k) = \theta^T \phi(k) \tag{11}$$

with

$$\theta^T = [\theta_1^T; \theta_2^T] = [r_1, \cdots, r_r, s_1, \cdots, s_r, h_1, \cdots, h_r, k_1, \cdots, k_r] \tag{11.a}$$

$$\begin{aligned}\phi(k)^T &= [\phi_1(k)^T; \phi_2(k)^T] \\ &= [Q_2 B^* y(k-1), \cdots, Q_2 B^* y(k-r), \\ &\quad Q_2 B^* u(k-1), \cdots, Q_2 B^* u(k-r), \\ &\quad Q_1 A^* y(k-1), \cdots, Q_1 A^* y(k-r), \\ &\quad Q_1 A^* u(k-1), \cdots, Q_1 A^* u(k-r)]\end{aligned} \tag{11.b}$$

$$y^*(k) = Q_1(q^{-1})Q_2(q^{-1})[B^*(q^{-1})u(k) - A^*(q^{-1})y(k)] \tag{11.c}$$

where θ_1 is a controller parameter vector, θ_2 is an auxiliary parameter vector, and $\phi_1(k)$ and $\phi_2(k)$ are measurement data vectors corresponding to θ_1 and θ_2, respectively.

For adaptive control strategy, let us introduce the following criterion function $J(k)$.

$$J(k) = \frac{1}{2} \sum_{j=1}^{k} \{\lambda^{k-j}[y^*(j) - \phi(j)^T \hat{\theta}(k)]\}^2 \tag{12}$$

where $\hat{\theta}(k)$ is the estimate of θ, and λ is a weighting factor such as $0 < \lambda < 1$.

From (12), we obtain the following well-known exponentially weighted least-squares algorithm [7]:

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \Gamma(\lambda, k)\phi(k)[y^*(k) - \phi(k)^T \hat{\theta}(k-1)] \tag{13}$$

$$\Gamma(\lambda, k)^{-1} = \lambda^2 \Gamma(\lambda, k-1) + \phi(k)\phi(k)^T \tag{14}$$

where

$$\Gamma(\lambda, k) = [\Phi(\lambda, k)\Phi(\lambda, k)^T]^{-1} \tag{14.a}$$

$$\Phi(\lambda, k) = [\lambda^{k-1}\phi(1), \lambda^{k-2}\phi(2), \cdots, \phi(k)]. \tag{14.b}$$

The control input $u(k)$ is then determined as follows:

$$Q_1(q^{-1})u(k) = \hat{R}(k, q^{-1})y(k) + \hat{S}(k, q^{-1})u(k) + Q_1(q^{-1})v(k) \tag{15}$$

where $\hat{R}(k, q^{-1})$ and $\hat{S}(k, q^{-1})$ are time-varying estimated polynomials of $R(q^{-1})$ and $S(q^{-1})$, respectively.

To guarantee the parameter convergence, the uniform persistent spanning under block invariant state feedback can be applied to this controller [5]. This note focuses on the problem of the exponential weighting in the aforementioned adaptive algorithm. In the next section, we will describe a scheme to weight all the past measurement data exponentially.

III. Conditions for Exponential Data Weighting

In this section, we consider the case where the most recent measurement data are assumed to be more informative than past measurement data and hence we exponentially discard old data. From (11.b), measurement data matrix (14.b) can be rewritten as follows:

$$\Phi(\lambda, k) = \begin{bmatrix} \lambda^{k-1} y_r(0), & \lambda^{k-2} y_r(1), & \cdots, & y_r(k-1) \\ 0, & \lambda^{k-2} y_r(0), & \cdots, & y_r(k-2) \\ \vdots & \ddots & & \vdots \\ 0, & \cdots & 0, & y_r(k-r) \\ \lambda^{k-1} u_r(0), & \lambda^{k-2} u_r(1), & \cdots, & u_r(k-1) \\ 0, & \lambda^{k-2} u_r(0), & \cdots, & u_r(k-2) \\ \vdots & \ddots & & \vdots \\ 0, & \cdots & 0, & u_r(k-r) \\ \lambda^{k-1} y_s(0), & \lambda^{k-2} y_s(1), & \cdots, & y_s(k-1) \\ 0, & \lambda^{k-2} y_s(0), & \cdots, & y_s(k-2) \\ \vdots & \ddots & & \vdots \\ 0, & \cdots & 0, & y_s(k-r) \\ \lambda^{k-1} u_s(0), & \lambda^{k-2} u_s(1), & \cdots, & u_s(k-1) \\ 0, & \lambda^{k-2} u_s(0), & \cdots, & u_s(k-2) \\ \vdots & \ddots & & \vdots \\ 0, & \cdots & 0, & u_s(k-r) \end{bmatrix}$$

where

$$y_r(k) = Q_2(q^{-1}) B^*(q^{-1}) y(k),$$
$$u_r(k) = Q_2(q^{-1}) B^*(q^{-1}) u(k)$$
$$y_s(k) = Q_1(q^{-1}) A^*(q^{-1}) y(k),$$
$$u_s(k) = Q_1(q^{-1}) A^*(q^{-1}) u(k).$$

To maintain exponential weighting properties for the past measurement data in the measurement data vectors, the coefficients of $Q_1(q^{-1})$ and $Q_2(q^{-1})$ are determined with zero initial conditions for $k < 0$ as follows.

Consider the third element $\lambda^{k-3} y_r(2)$ of the first row of (16)

$$\lambda^{k-3} y_r(2) = \lambda^{k-3}\left[\left(q_{21} b_1^* + b_2^*\right) y(0) + b_1^* y(1)\right]. \quad (17)$$

For exponential weighting, the coefficient q_{21} should be chosen as

$$q_{21} = \left(b_1^{*2} - b_2^*\right) / b_1^*. \quad (18)$$

By the same way, from the fourth element $\lambda^{k-3} y_r(3)$ of the first row of (16), the coefficient q_{22} should be chosen as

$$q_{22} = \left[\left(b_1^* b_2^* - b_3^*\right) + q_{21}\left(b_1^{*2} - b_2^*\right)\right] / b_1^*. \quad (19)$$

Therefore, the following recursive equation for the coefficients of $Q_2(q^{-1})$ may be obtained.

$$q_{2j} = \left[\sum_{i=0}^{j-1} q_{2i}\left(b_1^* b_{j-i}^* - b_{j-i+1}^*\right)\right] \Big/ b_1^*, \quad b_{r+1}^* = 0, \quad j = 1, 2, \cdots, r. \quad (20)$$

To determine the coefficients of $Q_1(q^{-1})$ for exponential data weighting as in $Q_2(q^{-1})$, consider the second element $\lambda^{k-2} y_s(1)$ of the $(2r+1)$th row of (16)

$$\lambda^{k-2} y_s(1) = \lambda^{k-2}\left[\left(q_{11} + a_1^*\right) y(0) + y(1)\right]. \quad (21)$$

To make exponential weighting, the coefficient q_{11} should be chosen as

$$q_{22} = f - a_1^* \quad (22)$$

where f is an arbitrary constant.

By the same way, from the third element $\lambda^{k-3} y_s(2)$ of the $(2r+1)$th row of (16), the coefficient q_{12} should be chosen as

$$q_{12} = f a_1^* - a_2^* + q_{11}\left(f - a_1^*\right). \quad (23)$$

Therefore, the following recursive equation for the coefficients of $Q_1(q^{-1})$ may obtained

$$q_{1j} = \sum_{i=0}^{j-1} q_{1i}\left(f a_{j-i-1}^* - a_{j-i}^*\right), \quad a_0^* = 1, \quad j = 1, 2, \cdots, r. \quad (24)$$

From (20) and (24), (16) can be rewritten as follows:

$$\Phi(\lambda, k) = \begin{bmatrix} 0, & \lambda^{k-2} b_1^* y(0), & \lambda^{k-3} b_1^*\{b_1^* y(0) + y(1)\}, & \lambda^{k-4} b_1^*\{(b_1^*)^2 y(0) + b_1^* y(1) + y(2)\}, & \cdots, & b_1^*\{(b_1^*)^{k-2} y(0) + \cdots + y(k-2)\} \\ \vdots & 0, & \lambda^{k-3} b_1^* y(0), & \lambda^{k-4} b_1^*\{b_1^* y(0) + y(1)\}, & \cdots, & b_1^*\{(b_1^*)^{k-3} y(0) + \cdots + y(k-3)\} \\ \vdots & \vdots & 0, & \lambda^{k-4} b_1^* y(0), & \cdots, & b_1^*\{(b_1^*)^{k-4} y(0) + \cdots + y(k-4)\} \\ \vdots & \vdots & \vdots & 0, & \cdots, & b_1^*\{(b_1^*)^{k-5} y(0) + \cdots + y(k-5)\} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \cdots, & b_1^*\{(b_1^*)^{k-r-1} y(0) + \cdots + y(k-r-1)\} \\ 0, & \lambda^{k-2} b_1^* u(0), & \lambda^{k-3} b_1^*\{b_1^* u(0) + u(1)\}, & \lambda^{k-4} b_1^*\{(b_1^*)^2 u(0) + b_1^* u(1) + u(2)\}, & \cdots, & b_1^*\{(b_1^*)^{k-2} u(0) + \cdots + u(k-2)\} \\ \vdots & 0, & \lambda^{k-3} b_1^* u(0), & \lambda^{k-4} b_1^*\{b_1^* u(0) + u(1)\}, & \cdots, & b_1^*\{(b_1^*)^{k-3} u(0) + \cdots + u(k-3)\} \\ \vdots & \vdots & 0, & \lambda^{k-4} b_1^* u(0), & \cdots, & b_1^*\{(b_1^*)^{k-4} u(0) + \cdots + u(k-4)\} \\ \vdots & \vdots & \vdots & 0, & \cdots, & b_1^*\{(b_1^*)^{k-5} u(0) + \cdots + u(k-5)\} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & 0, & 0, & \cdots, & b_1^*\{(b_1^*)^{k-r1} u(0) + \cdots + u(k-r-1)\} \\ \lambda^{k-1} y(0), & \lambda^{k-2}\{f y(0) + y(1)\}, & \lambda^{k-3}\{f^2 y(0) + f y(1) + y(2)\}, & \cdots & , & f^{k-1} y(0) + \cdots + y(k-1) \\ 0, & \lambda^{k-2} y(0), & \lambda^{k-3}\{f y(0) + y(1)\}, & \cdots & , & f^{k-2} y(0) + \cdots + y(k-2) \\ \vdots & 0, & \lambda^{k-3} y(0), & \cdots & , & f^{k-3} y(0) + \cdots + y(k-3) \end{bmatrix}$$

$$\begin{array}{lllll}
\vdots & \vdots & 0, & \cdots & , f^{k-4}y(0) + \cdots + y(k-4) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0, & 0, & 0, & \cdots & , f^{k-r}y(0) + \cdots + y(k-r) \\
\lambda^{k-1}u(0), & \lambda^{k-2}\{fu(0) + u(1)\}, & \lambda^{k-3}\{f^2u(0) + fu(1) + u(2)\}, & \cdots & , f^{k-1}u(0) + \cdots + u(k-1) \\
0, & \lambda^{k-2}u(0), & \lambda^{k-3}\{fu(0) + u(1)\}, & \cdots & , f^{k-2}u(0) + \cdots + u(k-2) \\
\vdots & 0, & \lambda^{k-3}u(0), & \cdots & , f^{k-3}u(0) + \cdots + u(k-3) \\
\vdots & \vdots & 0, & \cdots & , f^{k-4}u(0) + \cdots + u(k-4) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0, & 0, & 0, & \cdots & , f^{k-r}u(0) + \cdots + u(k-r)
\end{array} \Bigg] \quad (25)$$

As (25) shows, to weight the past measurement data exponentially, b_1^* and f should be chosen in the following range

$$0 < |b_1^*|, \; |f| \leq \lambda < 1. \quad (26)$$

It should be noted that the coefficients of $Q_1(q^{-1})$ and $Q_2(q^{-1})$ should be chosen to make the polynomials $Q_1(q^{-1})$ and $Q_2(q(^{-1})$ stable. The following proposition gives sufficient conditions to make the polynomials $Q_1(q^{-1})$ and $Q_2(q^{-1})$ stable in connection with the reference model.

Proposition 1: If the coefficients of $Q_1(q^{-1})$ and $Q_2(q^{-1})$ are determined from (24) and (20), respectively, and the coefficients of $A^*(q^{-1})$ and $B^*(q^{-1})$ of the reference model are given as follows:

$$a_i^* = \left(\frac{1}{\alpha}\right)^i, \quad i = 0, 1, 2, \cdots, r, \qquad \text{with } \alpha > 1 \quad (27)$$

$$b_i^* = \left(\frac{1}{\beta}\right)^{i-1} b_1^*, \quad i = 1, 2, \cdots, r, \qquad \text{with } \beta > 1 \quad (28)$$

then the coefficient b_1^* and the constant f should be chosen in the following ranges to make the polynomials $Q_1(q^{-1})$ and $Q_2(q^{-1})$ stable:

$$\frac{1}{\alpha} < f \leq \lambda < 1 \quad (29)$$

$$\frac{1}{\beta} < b_1^* \leq \lambda < 1 \quad \text{for sufficiently large } r. \quad (30)$$

Proof: See the Appendix.

If the measurement data in the measurement data vectors are not weighted exponentially, past measurement data on which should be imposed smaller weighting might be weighted greater than recent measurement data. This means that old data in the measurement data vectors are not exponentially discarded. In this case, the estimate of the parameter vector converges very slowly to its true value or does not converge to its true value. This scheme, however, always guarantees exponential weighting to all past measurement data. And this scheme would be more useful in the case of time-varying systems because greater weighting is attached to more recent measurement data.

IV. Conclusions

From pole and zero placement equations, a specially structured nonminimal model is derived. This model is used to design a direct model reference adaptive pole placement controller. To estimate the control parameters, an exponentially weighted least-squares algorithm is employed. To maintain the exponential weighting properties in direct model reference adaptive pole placement controller, a scheme of selecting the characteristic polynomials of the sensitivity function filters in connection with the reference model is proposed. Since all the past measurement data are weighted exponentially, this scheme guarantees fast convergence of all estimates.

Appendix

We will use the well-known monotony condition which guarantees all roots of the given polynomial lie inside the unit circle [8]. In (20)

$$\text{if } b_1^* > \frac{b_{j-i+1}^*}{b_{j-i}^*}, \quad \text{then } q_{2j} > 0, \qquad j = 0, 1, 2, \cdots, r. \quad (\text{A}.1)$$

From (20), we get

$$\begin{aligned}
q_{2j-1} - q_{2j} &= \left[\sum_{i=0}^{j-2} q_{2i}\left(b_1^* b_{j-i-1}^* - b_{j-i}^*\right) - \sum_{i=0}^{j-1} q_{2i}\left(b_1^* b_{j-i}^* - b_{j-i+1}^*\right) \right] \Big/ b_1^* \\
&= \sum_{i=0}^{j-2} q_{2i}\left[b_1^*\left(1 - b_1^{*2} + b_2^*\right) b_{j-i-1}^* - \left(1 + b_1^* - b_1^{*2} + b_2^*\right) b_{j-i}^* + b_{j-i+1}^* \right] / b_1^*. \quad (\text{A}.2)
\end{aligned}$$

If (A.1) is satisfied and (A.2) is positive, the inequality condition of the monotony condition is always satisfied.

For this purpose, if we choose

$$b_i^* = \left(\frac{1}{\beta}\right)^{i-1} b_1^*, \qquad i = 1, 2, \cdots, r \quad \text{with } \beta > 1 \quad (\text{A}.3)$$

then (A.1) is satisfied because

$$b_1^* > \frac{1}{\beta}. \quad (\text{A}.4)$$

And if r is sufficiently large, (A.2) becomes as follows:

$$q_{2j-1} - q_{2j} = \sum_{i=0}^{j-2} q_{2i} \left(\frac{1}{\beta}\right)^{j-i} \left[\left(b_1^* - b_1^{*3}\right)\beta^2 + \left(2b_1^{*2} - b_1^* - 1\right)\beta - b_1^* + 1 \right]. \quad (\text{A}.5)$$

Let the square bracket term in (A.5) be $f(\beta)$ as a function of β as follows:

$$f(\beta) = \left(b_1^* - b_1^{*3}\right)\beta^2 + \left(2b_1^{*2} - b_1^* - 1\right)\beta - b_1^* + 1. \quad (\text{A}.6)$$

Since $0 < b_1^* \leq \lambda < 1$, we get $(b_1^* - b_1^{*3}) > 0$, and the roots of $f(\beta)$ are $1/b_1^*$ and $1/(1 + b_1^*)$.

Since $\beta > 1$, to be $f(\beta) > 0$, β should be

$$\beta > 1/b_1^*. \tag{A.7}$$

If (A.7) is satisfied, (A.5) is always positive. (A.7) is the same condition as (A.4).

Therefore, from (26) and (A.7), we get

$$\frac{1}{\beta} < b_1^* \leq \lambda < 1.$$

By the same way, (29) can be obtained.

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Reduced-Order Dynamic Compensator Design for Stability Robustness of Linear Discrete-Time Systems

Sri R. Kolla and James B. Farison

Abstract—**This note presents a reduced-order dynamic compensator design with stability robustness for linear discrete systems, by including a stability robustness component in addition to the standard quadratic state and control terms in the performance criterion. The robustness component is based on a recently developed unstructured perturbation stability bound for time-varying perturbations. The controller design is developed by the parameter optimization technique and involves the solution of five algebraic matrix equations, four of which are discrete-time Lyapunov matrix equations.**

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S. Kolla was with the Department of Electrical Engineering, University of Toledo, Toledo, OH 43606. He is now with the Department of General Engineering, Pennsylvania State University, Sharon, PA 16146.

J. Farison is with the Department of Electrical Engineering, University of Toledo, Toledo, OH 43606.

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I. Introduction

The design of controllers for stability robustness of linear systems has been an active area of research for some time [1], [2]. In a recent survey paper, Siljak [1] reviewed several existing methods for robust control design. However, most of the methods he considered deal with continuous-time systems. Recently, robustness methods for discrete-time systems have also begun to receive considerable interest [3]–[17].

The robust controller design methods can be divided into two approaches: the frequency-domain [18] and the time-domain [1]. In the frequency domain, the robustness is measured either as gain and phase margins or the tolerance of plant perturbations [3]. A detailed account of these methods for continuous-time systems can be found in [18], [19]. For discrete-time full-state feedback linear quadratic (LQ) regulators, Shaked [4] investigated the stability margins. Recovering LQ robustness properties for state estimator-based linear quadratic Gaussian (LQG) regulators is addressed by Ishihara and Takeda [5]. Recently, Barratt and Boyd [3] investigated stability robustness of discrete-time LQG regulators for both gain and phase margins and tolerance of plant perturbations.

In the time-domain, robustness is usually measured by the tolerance of plant matrix perturbations. A survey of these methods for continuous-time systems can be found in [1]. Some of the existing time-domain controller design methods for discrete-time systems are applicable to "matched" uncertain systems [6]–[10]. Other controller design methods follow the approach of establishing bounds on allowable perturbations, and then using these bounds to select feedback controllers, such as LQ regulators [11]–[14]. However, these controller design methods do not directly consider the stability robustness aspect in the design itself. This paper describes a control design method that includes a stability robustness condition explicitly in the design procedure. This stability robustness condition is based on a recently developed unstructured perturbation bound on time-varying perturbations that used the Lyapunov approach [15]. Similar bounds using the Lyapunov approach have appeared elsewhere in the literature [13], [16], [17].

The form of the controller considered in this note is a reduced-order linear dynamic compensator which operates on the available outputs [20]–[23]. However, the general problem formulation is such that state and output feedback designs are special cases. The controller is designed using a parameter optimization procedure [20]. The problem formulation of the controller design in this note is similar to that of the continuous-time counterparts of Menga and Dorato [22] and Yedavalli [23]. However, the robustness constraint and the final solution are different from [22], [23]. A brief review of the stability robustness bound is presented in Section II. Section III gives the system description and the performance index specification. The control design by parameter optimization technique is described in Section IV. In Section V the control design method is specialized to the state feedback and output feedback cases. A simple example in Section VI illustrates the new results, with conclusions in Section VII.

II. Review of Stability Robustness Bound for Discrete Systems

In this section, we briefly review an unstructured perturbation bound for robust stability of linear discrete-time systems. The bound is derived using Lyapunov theory and singular value analysis [15], and is a sufficient condition.

Consider the linear discrete-time system described by the differ-